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On solutions of the compressible laminar boundary-layer equations and their behaviour near separation

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A numerical solution of the two-dimensional compressible laminar boundary-layer equations up to the point of separation is presented. For a particular mainstream velocity distribution it is necessary to specify the surface temperature (or the heat flux across the surface), the suction velocity, the free-stream Mach number and the viscosity–temperature relationship for a solution to be generated. The effect upon the position of separation of a hot or cold wall and of varying the free-stream Mach number is given special emphasis. The variations of the skin friction, heat transfer and various boundary-layer thicknesses for compressible flow past a circular cylinder and for flow with a linearly retarded mainstream were found. The behaviour of the solutions close to separation is investigated. Known functions which model the skin friction and heat transfer are introduced and are used to match the numerical solutions with the Buckmaster (1970) expansions.

1. Introduction

In recent years a great deal of interest has been centred on the behaviour of the boundary-layer equations near the point where the skin friction vanishes. At or very near this point the equations appear to develop a singularity and physically the boundary layer separates. The nature of the singularity in the incompressible case is well known. The early work of Goldstein (1948), with further developments by Stewartson (1958) and Terrill (1960), has shown that the skin friction has a square-root singularity at separation. Numerical investigations by Leigh (1955) and Terrill (1960) have succeeded in matching the numerical solution with the asymptotic expansion. Integration of the boundary-layer equations beyond the point of vanishing skin friction has not yet proved possible except when a small bubble is allowed to develop, with a subsequent reattachment so that the more catastrophic breakaway does not occur (Catherall & Mangler 1966). The most recent developments have centred on free-streamline approaches and are reviewed by Stewartson (1975).

The nature of the singularity in the compressible case was first studied by Stewartson (1962), who found that in order to satisfy a certain integral condition there are two alternatives. Either the heat transfer vanishes and the singularity is the same as in the incompressible case, or the heat transfer is non-vanishing and the skin friction is regular. When an integral condition could not be satisfied in the incompressible case Stewartson (1958) found that the introduction of a logarithmic term resolved the difficulty. In this case the inclusion of a logarithmic term proved to be inconclusive. Numerical evidence in support of these alternatives was scarce. Poots (1960) had solved the equations but could not compute precise results close to separation owing to a lack of computing power. Williams, quoted in Brown & Stewartson (1969), had found evidence of singular behaviour for both hot and cold walls. Also Merkin (1969) studied a free convection problem which exhibited separation and he found that for a heated wall the skin friction still possesses a singularity but the heat transfer does not vanish. In the light of these results Buckmaster (1970) re-examined the equations and found that the integral condition could be satisfied without the restriction of vanishing heat transfer provided log (log) terms were included in the Stewartson expansion. However, a further integral condition appeared to restrict the validity of the expansion to the cold-wall case only. Subsequent numerical work by Werle & Senechal (1973) on separating supersonic boundary-layer flows with linearly and quadratically retarded mainstreams has supported the Buckmaster expansion in the cold-wall case but has suggested the presence of weak singular behaviour of the skin friction for the hot-wall case. However, some results which did not support the Buckmaster expansion were presented by Wilks (1974), who considered a separating flow in free convection about a semi-infinite flat plate with a prescribed heat flux at the wall. These results suggested that the singularity was that of a three-fifths power and thus pointed to a re-examination of the original transformation used by Goldstein (1930, 1948).

It was thus thought that further numerical evidence was necessary and so a method of solution has been developed which can be used to solve the equations up to the point of separation with various conditions prescribed, including that of suction through a porous wall. The equations are first transformed using part of the Illingworth–Stewartson transformation but the co-ordinate along the wall is left untransformed so that the equations do not take the usual incompressible form which leads to the study of similar solutions (Stewartson 1949; Cohen & Reshotho 1956). Instead, the equations are made non-dimensional and transformed further in such a way as to make them more amenable to numerical solution. This transformation also has the advantage of allowing the mainstream velocity to be specified in the untransformed co-ordinates, which means that more realistic models can be studied.

Two particular mainstream velocity distributions have been studied. The first is flow with a linearly retarded mainstream, which provides a situation where there is an adverse pressure gradient throughout the boundary layer. The second is compressible flow past a circular cylinder, where the irrotational mainstream solution obtained by Simasaki (1956) has been used. In the latter case the external velocity is dependent on the free-stream Mach number. In both cases various constant wall temperatures and free-stream Mach numbers were specified for a model fluid whose Prandtl number $\sigma = 1$ and whose viscosity is proportional to the absolute temperature.

The transformation of the equations and the numerical scheme are only outlined here. A complete account can be found in Davies (1975) together with all the numerical results obtained in the study. As a check on the numerical method the known results of other authors were reproduced. In particular, a modification of the program used enabled a check to be made on the results obtained by Merkin (1969) and Wilks (1974). Merkin's results were reproduced and more accurate results were obtained by reducing the step lengths. Wilks' results appear to be in error as they could be reproduced only by modelling a derivative boundary condition by using a forward difference. Wilks (private communication) has indicated that a central difference approximation was actually used. However, a more accurate boundary condition can be obtained by using a Taylor-series expansion near the wall and the familiar square-root singularity near separation was then obtained.

Finally, an attempt has been made to use the numerical results to examine the behaviour of the solutions close to the point of vanishing skin friction. The method employed is to model the expansions for the skin friction and the heat transfer in terms of quantities which can be obtained from the numerical results. This is achieved by the introduction of three known functions whose behaviour as separation is approached can be used to provide guidance and re-assurance as to the nature of the expansions at each stage. It is found tht the expansion obtained by Buckmaster (1970) can be fitted to the numerical results not only for the cold-wall case but also for a hot wall.

2. The equations

In the usual notation, for perfect fluids, the equations of motion for a steady, two-dimensional, compressible boundary-layer flow are

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0, \tag{1}$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y} = \rho_1 u_1\frac{du_1}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right). \tag{2}$$

The energy equation is

$$\rho C_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)+\rho_1 uu_1\frac{du_1}{dx} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2. \tag{3}$$

The viscosity μ is temperature dependent and using Chapman's viscosity we have $\mu = C(x)\,T$, where $C(x)$ is a suitable function of x.

The surface over which the boundary layer flows is maintained at some arbitrary temperature $T_w(x)$. Further, if the effect of removing fluid through the surface is taken into account a suction velocity distribution is prescribed. The boundary conditions for the above equations are therefore

$$u = 0, \quad v = -(u_\infty \nu_\infty/l)^{\frac{1}{2}}\, v_s(x), \quad T = T_w(x) \quad \text{at} \quad y = 0,\ x \geqslant 0; \tag{4}$$

$$u = u_1, \quad T = T_1 \quad \text{as} \quad y \to \infty,\ x \geqslant 0, \tag{5}$$

where $v_s(x)$ is the non-dimensional velocity of suction. Application of the Illingworth–Stewartson transformation reduces the equations of motion to a form similar to those for the incompressible boundary layer. However, the transformation of the x co-ordinate has practical limitations and we omit this part of the transformation. The required transformations are

$$\rho u = \rho_\infty \partial\psi/\partial y, \quad \rho v = -\rho_\infty \partial\psi/\partial x, \tag{6a}$$

$$Y = \frac{a_1}{a_\infty}\int_0^y \frac{\rho}{\rho_\infty}\,dy, \tag{6b}$$

$$\frac{T}{T_1} = 1 + \frac{T_\infty}{T_1}\left[1+\frac{\gamma-1}{2}M_\infty^2\right]S+\frac{\gamma-1}{2a_1^2}(u_1^2-u^2), \tag{6c}$$

together with the assumption that the mainstream is homenergic. The resulting equations are then made non-dimensional by writing

$$x' = \frac{x}{l}, \quad Y' = \frac{R^{\frac{1}{2}}Y}{l}, \quad \psi' = \frac{R^{\frac{1}{2}}\psi}{u_\infty l}, \quad u' = \frac{u}{u_\infty}, \quad U_1' = \frac{U_1}{u_\infty}, \tag{7}$$

where $R = u_\infty l/\nu_\infty$ is the Reynolds number and l is a representative length. Hence (with the dashes omitted)

$$\frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial x\partial Y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial Y^2} = U_1\frac{dU_1}{dx}(1+S) + \left(\frac{a_1}{a_\infty}\right)^{(3\gamma-1)/(\gamma-1)} C_1\frac{\partial^3\psi}{\partial Y^3}, \tag{8}$$

$$\frac{\partial\psi}{\partial Y}\frac{\partial S}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial S}{\partial Y} = \left(\frac{a_1}{a_\infty}\right)^{(3\gamma-1)/(\gamma-1)} \frac{1}{C_1\sigma}\left\{\frac{\partial^2 S}{\partial Y^2} + (\sigma-1)H_\infty\left(\frac{a_1}{a_\infty}\right)^2\frac{\partial^2}{\partial Y^2}\left(\frac{\partial\psi}{\partial Y}\right)^2\right\}, \tag{9}$$

where $\sigma = \mu C_p/k$ is the Prandtl number, which is assumed to be constant,

$$U_1 = a_\infty u_1/a_1, \quad C_1 = \mu_\infty/C(x)\,T_\infty$$

and

$$H_\infty = \tfrac{1}{2}(\gamma-1)M_\infty^2/[1+\tfrac{1}{2}(\gamma-1)M_\infty^2].$$

The boundary conditions are

$$\left.\begin{aligned} &\partial\psi/\partial Y = 0, \quad S = S_w(x) \\ &\frac{\partial\psi}{\partial x} = \left(\frac{a_1}{a_\infty}\right)^{2\gamma/(\gamma-1)} v_s(x)\Big/\left[1+\frac{(\gamma-1)}{2}M_\infty^2\right](1+S_w) \end{aligned}\right\} \quad \text{at} \quad Y = 0, \quad x \geqslant 0, \tag{10}$$

$$\partial\psi/\partial Y \to (a_\infty/a_1)\,u_1, \quad S \to 0 \quad \text{as} \quad Y\to\infty, \quad x \geqslant 0. \tag{11}$$

The equations are now transformed to a form more amenable to numerical solution. For the incompressible boundary-layer equations, Terrill (1960) used a transformation due to Görtler. We shall use a modified form of this transformation here. Write

$$\xi = \int_0^x u_1\,dx, \quad \eta = \frac{A(x)\,Y}{(2\xi)^{\frac{1}{2}}}, \quad \psi = (2\xi)^{\frac{1}{2}}\phi(\xi,\eta), \tag{12}$$

where

$$A(x) = (a_\infty/a_1)^{(3\gamma-1)/(\gamma-1)}\,u_1(x)\,C_1(x),$$

and obtain

$$\frac{\partial^3\phi}{\partial\eta^3} + \phi\frac{\partial^2\phi}{\partial\eta^2} + \beta(\xi)\left\{G(\xi)(1+S) - \left(\frac{\partial\phi}{\partial\eta}\right)^2\right\} = 2\xi\left(\frac{\partial\phi}{\partial\eta}\frac{\partial^2\phi}{\partial\xi\partial\eta} - \frac{\partial\phi}{\partial\xi}\frac{\partial^2\phi}{\partial\eta^2}\right), \tag{13}$$

$$\frac{1}{\sigma}\frac{\partial^2 S}{\partial\eta^2} + \phi\frac{\partial S}{\partial\eta} = 2\xi\left\{\frac{\partial\phi}{\partial\eta}\frac{\partial S}{\partial\xi} - \frac{\partial\phi}{\partial\xi}\frac{\partial S}{\partial\eta}\right\} + \frac{(1-\sigma)}{\sigma}H(\xi)\frac{\partial^2}{\partial\eta^2}\left(\frac{\partial\phi}{\partial\eta}\right)^2, \tag{14}$$

where

$$\beta(\xi) = \frac{2\xi\,dA/dx}{u_1(x)\,A(x)}, \quad G(\xi) = \frac{U_1\,dU_1/dx}{A\,dA/dx}$$

and

$$H(\xi) = (a_\infty/a_1)^{4\gamma/(\gamma-1)}C_1^2 H_\infty u_1^2.$$

The boundary conditions are

$$\partial\phi/\partial\eta = 0, \quad S = S_w(x) \quad \text{at} \quad \eta = 0, \quad \xi \geqslant 0, \tag{15}$$

$$\phi(\xi,0) + 2\xi\frac{\partial\phi}{\partial\xi}(\xi,0) = (2\xi)^{\frac{1}{2}}\left(\frac{a_1}{a_\infty}\right)^{2\gamma/(\gamma-1)} v_s(x)\Big/u_1(x)\left[1+\frac{(\gamma-1)}{2}M_\infty^2\right](1+S_w) = K_s(\xi), \tag{16}$$

where $K_s(\xi)$ is the non-dimensional velocity of suction, and

$$\frac{\partial \phi}{\partial \eta} \to \frac{1}{C_1}\left(\frac{a_1}{a_\infty}\right)^{2\gamma/(\gamma-1)}, \quad S \to 0 \quad \text{as} \quad \eta \to \infty, \ \xi \geqslant 0. \tag{17}$$

The equations at the leading edge $\xi = 0$ are given by setting $\xi = 0$ and replacing partial derivatives with respect to η by full derivatives. Thus, for a particular mainstream velocity distribution we need specify only $C_1(x)$, M_∞, $v_s(x)$ and $S_w(x)$ for a solution of the equations to be undertaken. The non-dimensional skin friction is given by

$$\tau_w = C_1 \left(\frac{a_\infty}{a_1}\right)^{2\gamma/(\gamma-1)} \frac{u_1^2(x)}{(2\xi)^{\frac{1}{2}}} \left(\frac{\partial^2 \phi}{\partial \eta^2}\right)_{\eta=0} \tag{18}$$

and the non-dimensional heat transfer is given by

$$Q_s = \frac{-C_1(x)}{(1+S_w)} \frac{u_1(x)}{(2\xi)^{\frac{1}{2}}} \left(\frac{\partial S}{\partial \eta}\right)_{\eta=0}. \tag{19}$$

Expressions for the displacement thickness δ_1, the momentum thickness δ_2 and the enthalpy thickness δ_T follow similarly.

3. The numerical solution

The equations are solved numerically by using the Hartree–Wormersly method. Derivatives in the ξ direction are replaced by differences and all other quantities by averages. We write $q = \partial\phi/\partial\eta$. Given q_1 and S_1 at a particular cross-section ξ_1 we can find q_2 and S_2 at ξ_2. With $v = q_1 + q_2$, $\theta = S_1 + S_2$ and, by defining a suitable iterative procedure, the equations become

$$\mathbf{A}^{(m)}\mathbf{v}^{(m+1)} = \mathbf{C}^{(m)}, \quad \mathbf{B}^{(m)}\boldsymbol{\theta}^{(m+1)} = \mathbf{D}^{(m)}, \tag{20), (21}$$

where $\mathbf{A}^{(m)}$ is a special matrix described by Terrill (1960) and $\mathbf{B}^{(m)}$ is a tridiagonal matrix.

The initial profiles were found by solving (13) and (14) numerically with $\xi = 0$. However it is not possible to take $\xi = 0$ as the starting point for the solution as the above procedure does not converge there. The integration was started at $\xi = 10^{-6}$ with an initial step of 5×10^{-7}. The step length was increased as the distance away from the leading edge increased.

In the η direction a step length $h = 0{\cdot}05$ was taken over the range $\eta = 0$–10. The step length was then halved and the results from the two integrations were compared to ensure that the difference between the two solutions was less than 5×10^{-5}. In the ξ direction the solution from ξ_1 to ξ_2 was first obtained in one step and then in two steps. These results were likewise compared to ensure that the difference between the two solutions was less than 5×10^{-5}.

4. The numerical results

The above method has been applied to two different mainstream velocity distributions: those for flow with a linearly retarded mainstream and for compressible flow past a cylinder. Most results have been obtained for a model fluid with $\sigma = 1$ and $C_1(x) = 1$. The adiabatic index γ has been taken to be equal to $1{\cdot}4$ throughout.

The main intention has been to study the effect of heat transfer on separation and for this purpose a constant wall temperature and zero suction have been specified. To simulate a hot wall we have taken $S_w = 1$ and for a cold wall $S_w = -\frac{1}{2}$. Thus, for example, when $M_\infty = 0$, $S_w = 1$ signifies a wall temperature twice that of the free stream whilst $S_w = -\frac{1}{2}$ gives a wall temperature half that of the free stream. Results were also obtained with $S_w = 0$, which is the case of an insulated wall with solution $S = 0$, and for $T_w = T_\infty$ for non-zero Mach number, with S_w chosen accordingly. The skin friction, the heat transfer at the wall and the various thicknesses have been calculated.† It is expected that heating the wall will cause separation to take place earlier whilst cooling delays separation. This is because, in this model, viscosity increases with temperature and thus viscous forces, which enhance the retarding effect of the pressure gradient, are greater for a heated wall.

Flow with a linearly retarded mainstream

The mainstream velocity distribution is given by $U_1/U_\infty = 1-\frac{1}{8}x$. Results were obtained for Mach numbers ranging from 0 to 1 for various S_w. Two tables of results for a hot wall ($S_w = 1$) and for a cold wall ($S_w = -\frac{1}{2}$) respectively with $M_\infty = 1$ are presented as examples.

Consider $M_\infty = 0$, so that dissipative heating does not occur. $S_w = 0$ corresponds to the incompressible case and separation is predicted to occur at $x_s = 0{\cdot}9583$. Leigh (1955) reports $x_s = 0{\cdot}9585$. When $S_w = 1$ the wall temperature is twice that of the free stream and the skin friction is reduced, thereby moving the position of separation upstream to $x_s = 0{\cdot}5889$. Conversely, cooling increases the skin friction and delays separation, so that, for $S_w = -\frac{1}{2}$, $x_s = 1{\cdot}4031$. Furthermore, it was found that the displacement thickness is increased by heating the wall whilst the momentum thickness is decreased since the density decreases with temperature. The opposite effect occurs when the wall is cooled.

If S_w is held constant and M_∞ is increased the position of separation moves upstream since under our transformation the wall temperature is increased. When $M_\infty = 1$ and $S_w = 0$, $x_s = 0{\cdot}8812$. For a hot wall ($S_w = 1$) $x_s = 0{\cdot}5240$ and for a cold wall ($S_w = -\frac{1}{2}$) $x_s = 1{\cdot}3305$. In the cold-wall case it was found that the skin friction increases with M_∞ at first but separation still takes place earlier. For $M_\infty = 0$ the wall is colder and the lower viscosity very near the wall may result in a lower skin-friction coefficient initially.

Compressible flow past a circular cylinder

The mainstream velocity distribution for the irrotational flow of a compressible fluid past a circular cylinder has been obtained by Simasaki (1956). It consists of a power series in M_∞ which converges for $M_\infty \leqslant 0{\cdot}4$. Results have thus been obtained for values of M_∞ ranging from 0 to 0·4 for various S_w.

The pressure gradient is favourable for $0 \leqslant x \leqslant \frac{1}{2}\pi$, so that the fluid is accelerated. For $x > \frac{1}{2}\pi$ the pressure gradient is adverse and separation occurs. It is found that for a particular S_w the skin friction has a behaviour similar to that of the pressure gradient. In all cases the effect of increasing M_∞ is to move the position of separation upstream since the pressure gradient becomes more adverse.

† Tables of these results for both types of mainstream flow can be obtained on request from the Editorial Office of the Journal or directly from the authors.

x	τ_w	Q_s	δ_1	δ_2	$-\delta_T$	S_{L1}	S_{L2}	$-H_L$	ξ
0·0000015	271·12	135·56	0·0052	0·0008	0·0012				
0·0001200	30·308	15·156	0·0467	0·0072	0·0110				
0·01030	3·22871	1·63142	0·4340	0·0676	0·1019				
0·1	0·91315	0·51031	1·3932	0·2152	0·3139				
0·2	0·54172	0·34795	2·0497	0·3121	0·4380				
0·3	0·34884	0·27045	2·6377	0·3925	0·5287				
0·4	0·20821	0·21741	3·2573	0·4661	0·6007				
0·5	0·06936	0·16473	4·1051	0·5368	0·6586				
0·5237	0·00607	0·13110	4·5856	0·5535	0·6698	0·787	0·0817	0·3140	0·1039
0·52376	0·00531	0·13035	4·5916	0·5536	0·6699	0·779	0·0761	0·3122	0·0977
0·52386	0·00396	0·12889	4·6024	0·5536	0·6699	0·764	0·0651	0·3087	0·0852
0·52397	0·00198	0·12627	4·6183	0·5537	0·6700	0·733	0·0450	0·3025	0·0615
0·523984	0·00144	0·12537	4·6226	0·5537	0·6700	0·719	0·0381	0·3003	0·0530
0·523997	0·00088	0·12424	4·6271	0·5537	0·6700	0·70	0·0293	0·2976	0·0419
0·5240011	0·00060	0·12355	4·6294	0·5537	0·6700	0·68	0·0219	0·2953	0·0323
0·5240046	0·00020	0·12223	4·6326	0·5537	0·6700	0·64	0·0135	0·2928	0·0210
0·5240050	0·00009	0·12166	4·6335	0·5537	0·6700	0·61	0·0089	0·2914	0·0146
0·52400505	0·00004	0·12125	4·6339	0·5537	0·6700	0·6	0·0056	0·2904	0·0093
0·52400507	0·00000	0·12089	4·6342	0·5537	0·6700	—	—	0·2896	0

TABLE 1. Retarded mainstream with $S_w = 1$, $M_\infty = 1$.

x	τ_w	$-Q_s$	δ_1	δ_2	δ_T	S_{L1}	S_{L2}	H_L	ξ
0·0000015	271·12	271·12	0·0014	0·0008	0·0002				
0·0001200	30·311	30·313	0·0128	0·0073	0·0021				
0·1	0·99809	1·0421	0·3758	0·2123	0·0604				
0·2	0·66792	0·73020	0·5429	0·3038	0·0860				
0·3	0·51350	0·58991	0·6804	0·3768	0·1060				
0·4	0·41621	0·50456	0·8058	0·4408	0·1231				
0·5	0·34587	0·44477	0·9261	0·4996	0·1385				
0·6	0·29068	0·39913	1·0457	0·5553	0·1525				
0·7	0·24489	0·36211	1·1680	0·6090	0·1655				
0·8	0·20525	0·33060	1·2960	0·6616	0·1777				
0·9	0·16965	0·30257	1·4337	0·7138	0·1892				
1·0	0·13653	0·27650	1·5861	0·7660	0·1999				
1·1014	0·10395	0·25045	1·7644	0·8196	0·2101				
1·2019	0·07049	0·22245	1·9834	0·8737	0·2193				
1·3039	0·02760	0·18089	2·3236	0·9304	0·2273				
1·32962	0·00443	0·14623	2·5283	0·9451	0·2289	1·359	0·1386	0·6322	0·1020
1·32993	0·00352	0·14392	2·5362	0·9453	0·2290	1·355	0·1233	0·6223	0·0910
1·33008	0·00297	0·14238	2·5410	0·9454	0·2290	1·354	0·1133	0·6156	0·0837
1·33035	0·00162	0·13790	2·5523	0·9455	0·2290	1·357	0·0837	0·5962	0·0617
1·330408	0·00114	0·13589	2·5568	0·9456	0·2290	1·36	0·0704	0·5875	0·0517
1·330437	0·00079	0·13418	2·5597	0·9456	0·2290	1·36	0·0586	0·5801	0·0430
1·330456	0·00043	0·13189	2·5628	0·9456	0·2290	1·4	0·0438	0·5702	0·0313
1·3304621	0·00020	0·12986	2·5647	0·9456	0·2290	1·5	0·0305	0·5615	0·0210
1·3304635	0·00006	0·12798	2·5656	0·9456	0·2290	1·7	0·0187	0·5533	0·0109
1·33046358	0·00003	0·12725	2·5661	0·9456	0·2290	—	—	0·5502	0

TABLE 2. Retarded mainstream with $S_w = -\frac{1}{2}$, $M_\infty = 1$.

In incompressible flow ($M_\infty = 0$, $S_w = 0$) separation is predicted to occur at

$$x_s = 1{\cdot}8231 \quad (104{\cdot}45^\circ).$$

Terrill (1960) obtained $x_s = 1{\cdot}8230$. As expected, heating promotes separation whilst cooling delays separation. When $S_w = 1$, $x_s = 1{\cdot}7591$ ($100{\cdot}79^\circ$) and, when $S_w = -\frac{1}{2}$, $x_s = 1{\cdot}9072$ ($109{\cdot}27^\circ$). In the accelerated region only viscous forces oppose the motion and a consequence of the increase in viscosity brought about by heating is that the skin friction increases, whereas it decreases if the wall is cooled. Since density decreases with temperature a hot (cold) wall results in a larger (smaller) displacement thickness and a smaller (larger) momentum thickness.

In incompressible flow with $M_\infty = 0{\cdot}4$ the position of separation for an insulated wall ($S_w = 0$) is $x_s = 1{\cdot}7327$ ($99{\cdot}27^\circ$). When $S_w = 1$, $x_s = 1{\cdot}6953$ ($97{\cdot}13^\circ$) and, when $S_w = -\frac{1}{2}$, $x_s = 1{\cdot}7892$ ($102{\cdot}51^\circ$).

When the cylinder is heated overshoot occurs in the accelerated region. The fluid in the outer region of the boundary layer is hardly affected by the retarding influence of the wall and is less dense than the fluid in the mainstream. The acceleration caused by the pressure gradient results in the fluid moving faster in this outer region than the fluid in the mainstream. Overshoot does not disappear completely in the retarded region since the forces do not have time to reduce the velocity (in the outer region of the boundary layer) sufficiently before separation occurs.

5. The behaviour of the solutions close to separation

In order to use the numerical results to investigate the behaviour of the skin friction and heat transfer close to separation it is necessary to develop the Buckmaster (1970) expansions in terms of the non-transformed, x co-ordinate. As in §4 we shall assume that $\sigma = 1$ and $C(x) = \mu_\infty/T_\infty$. Suction will not be considered as it has been shown that its effect is one of degree (Terrill 1960) and that it does not affect the nature of the singularity. We take as our starting point (8) and (9), which become

$$\frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial x\,\partial Y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial Y^2} = U_1\frac{dU_1}{dx}(1+S) + \left(\frac{a_1}{a_\infty}\right)^{(3\gamma-1)/(\gamma-1)}\nu_\infty\frac{\partial^3\psi}{\partial Y^3}, \tag{22}$$

$$\frac{\partial\psi}{\partial Y}\frac{\partial S}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial S}{\partial Y} = \left(\frac{a_1}{a_\infty}\right)^{(3\gamma-1)/(\gamma-1)}\nu_\infty\frac{\partial^2 S}{\partial Y^2}. \tag{23}$$

The boundary conditions are

$$\left.\begin{aligned} &\psi = \partial\psi/\partial Y = 0, \quad S = S_w(x) \quad \text{at} \quad Y = 0, \quad x \geqslant 0, \\ &\partial\psi/\partial Y \to (a_\infty/a_1)\,u_1, \quad S \to 0 \quad \text{as} \quad Y \to \infty, \quad x \geqslant 0. \end{aligned}\right\} \tag{24}$$

The equations are first made non-dimensional and then transformed in a way similar to the methods employed by Goldstein (1948) and Stewartson (1962). We take

$$x' = \frac{x_s - x}{l_s}, \quad Y' = \frac{R_s^{\frac{1}{2}}Y}{l_s}, \quad \psi' = \frac{R_s^{\frac{1}{2}}\psi}{l_s u_{1s}}, \quad U_1' = \frac{U_1}{u_{1s}}, \tag{25}$$

where

$$l_s = \frac{-u_{1s}}{(du_1/dx)_s}\left(\frac{a_\infty}{a_1}\right)_s^{2(\gamma+1)/(\gamma-1)}\frac{1}{(1+S_{ws})\,[1+\frac{1}{2}(\gamma-1)\,M_\infty^2]},$$

$R_s = u_{1s} l_s / \nu_\infty$ is a Reynolds number and the suffixes s denote values at separation. The resulting equations are then transformed by taking

$$\left.\begin{aligned} \xi = (x')^{\frac{1}{4}}, \quad \eta = \left(\frac{a_\infty}{a_1}\right)^{(3\gamma-1)/(\gamma-1)} Y'/2^{\frac{1}{2}}\xi, \quad \psi' = 2^{\frac{3}{2}}\xi^3 f(\xi,\eta),\\ S = S_{ws} + (1+S_{ws})\, g(\xi,\eta). \end{aligned}\right\} \tag{26}$$

Near $x' = 0$ we assume that both the pressure gradient and the heat transfer at the wall are regular functions. Thus

$$U_1' \frac{dU_1'}{dx'} = \left(U_1' \frac{dU_1'}{dx}\right)_s (1 + P_1 \xi^4 + P_2 \xi^8 + \ldots) \tag{27}$$

and

$$g(\xi, 0) = \xi^4 S_1 + \xi^8 S_2 + \ldots, \tag{28}$$

where the P's and S's are known. We find that

$$\frac{\partial^3 f}{\partial \eta^3} - 3f\frac{\partial^2 f}{\partial \eta^2} + 2\left(\frac{\partial f}{\partial \eta}\right)^2 + \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta} - \frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial \xi}\right) = (1+g)\sum_{r=0}^{\infty} P_r \xi^{4r} - \left(\frac{\partial f}{\partial \eta}\right)^2 \sum_{r=1}^{\infty} Q_r \xi^{4r}, \tag{29}$$

$$\frac{\partial^2 g}{\partial \eta^2} - 3f\frac{\partial g}{\partial \eta} + \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial g}{\partial \xi} - \frac{\partial g}{\partial \eta}\frac{\partial f}{\partial \xi}\right) = 0, \tag{30}$$

where $P_0 = 1$ and the Q's are known.

The boundary conditions are

$$f(\xi, 0) = \frac{\partial f}{\partial \eta}(\xi, 0) = 0, \quad g(\xi, 0) = \sum_{r=1}^{\infty} \xi^{4r} S_r. \tag{31}$$

We also have the requirement that neither f nor g is exponentially large as $\eta \to \infty$.

From now on we shall take S_w = constant since the effect of variable S_w is one of degree and will not influence the nature of the singularity. Thus $S_1 = S_2 = \ldots = 0$. Under the above transformations we also have

$$u = \left(\frac{a_\infty}{a_1}\right)^{2\gamma/(\gamma-1)} 2u_{1s}\xi^2 \frac{\partial f}{\partial \eta}, \tag{32}$$

$$\tau_w = S_c \left(\frac{a_\infty}{a_1}\right)^{2\gamma/(\gamma-1)} \xi \left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\eta=0}, \quad Q_s = \frac{-H_c}{\xi}\left(\frac{\partial g}{\partial \eta}\right)_{\eta=0}, \tag{33), (34}$$

where

$$H_c = \left[1 + \frac{(\gamma-1)}{2} M_\infty^2\right]^{\frac{1}{2}} (1+S_w)^{\frac{1}{2}} \left(\frac{a_1}{a_\infty}\right)_s^{(\gamma+1)/(\gamma-1)} \left(-\frac{du_1'}{dx'}\right)_s^{\frac{1}{4}} \frac{1}{2^{\frac{1}{4}}}$$

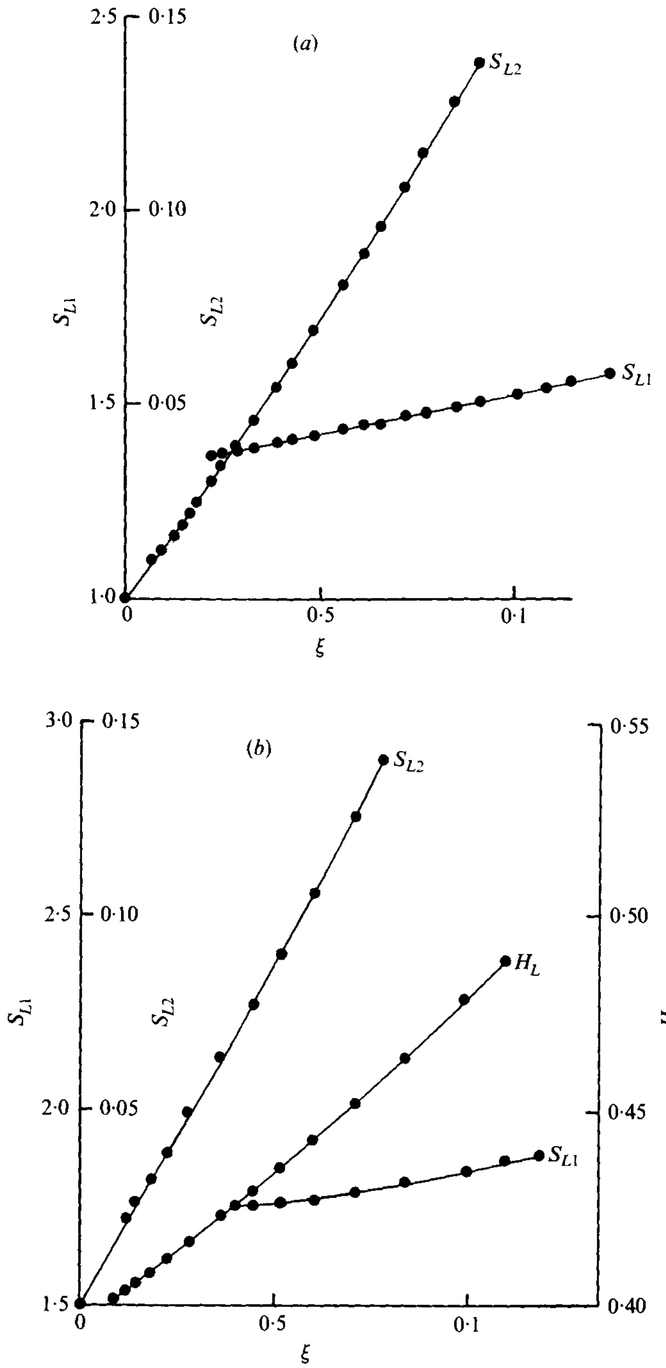
and $S_c = 2u_{1s}' H_c$ are constants. τ_w and Q_s are the non-dimensional skin-friction and heat-transfer coefficients respectively.

Following Stewartson (1962) and Buckmaster (1970) we look for solutions of the form

$$f(\xi,\eta) = \sum_{n=0}^{\infty} f_n(\xi,\eta)\xi^n, \quad g(\xi,\eta) = \sum_{n=0}^{\infty} g_n(\xi,\eta)\xi^n.$$

The ξ dependence of f_n and g_n may be logarithmic. Substituting for f and g in (29) and (30) we obtain a system of equations for the f_n and g_n with boundary conditions

$$f_n(0) = f_n'(0) = 0, \quad g_n(0) = 0, \quad n = 0, 1, 2, \ldots.$$



FIGURES 1 (*a*, *b*). For legend see facing page.

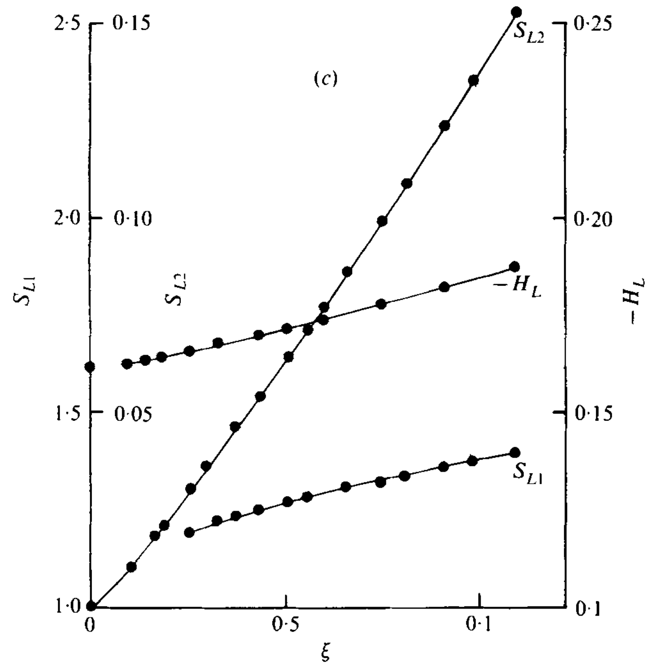


FIGURE 1. S_{L1}, S_{L2} and H_L for flow past a circular cylinder at $M_\infty = 0$. (*a*) $S_w = 0$, (*b*) $S_w = -\frac{1}{2}$, (*c*) $S_w = 1$.

Assuming that f_0 and g_0 are independent of ξ gives

$$g_0'' - 3f_0 g_0' = 0, \quad f_0''' - 3f_0 f_0'' + 2f_0'^2 = 1 + g_0.$$

From the expressions (32) and (33) for the velocity profile and skin friction respectively we must have $f_0', f_0'' \geqslant 0$ since just prior to separation the flow is forward and the skin friction is positive. Thus $f_0 \geqslant 0$ and, if $g_0'(0) \neq 0$, $g_0' \to \infty$ exponentially, which is not allowed. Hence we must have

$$g_0 = 0, \quad f_0 = \tfrac{1}{6}\eta^3.$$

Before continuing with the solutions for f_1 and g_1 some idea of their behaviour can be obtained by studying the numerical results near separation. We can write

$$S_{L1} = \frac{1}{\xi^2 S_c}\left(\frac{a_1}{a_\infty}\right)^{2\gamma/(\gamma-1)} \tau_w = f_1''(\xi, 0) + \xi f_2''(\xi, 0) + \ldots, \tag{35}$$

$$S_{L2} = \xi S_{L1} = \xi f_1''(\xi, 0) + \xi^2 f_2''(\xi, 0) + \ldots \tag{36}$$

and

$$H_L = -Q_s/H_c = g_1'(\xi, 0) + \xi g_2'(\xi, 0) + \ldots. \tag{37}$$

τ_w and Q_s have been obtained numerically. Once the position of separation x_s has been determined it is possible to calculate l_s, H_c and S_c. For a particular value of x we can calculate ξ, S_{L1}, S_{L2} and H_L. The functions S_{L1}, H_L and S_{L2} have been tabulated near $\xi = 0$ in tables 1 and 2 and profiles have been drawn in figures 1 (*a*), (*b*) and (*c*) for flow past a circular cylinder with $M_\infty = 0$ and $S_w = 0$, $-\frac{1}{2}$ and 1 respectively. For $S_w = 1$ the modulus of H_L has been drawn. When such curves are drawn for other values of M_∞ given in the tables the behaviour of the curves remains unchanged. Also the curves for a linearly retarded mainstream exhibit the same behaviour.

Near $\xi = 0$, S_{L1} and S_{L2} are sensitive to the values of τ_w and x. Changes in the fifth decimal place for τ_w or in the eighth decimal place for x can have a marked effect on the values of S_{L1} (in particular) and S_{L2}. As a result S_{L1} could be calculated to only one decimal place very near $\xi = 0$ and even so the last figure is unreliable. However, reliable bounds for S_{L1} near $\xi = 0$ have been obtained by allowing for rounding errors in both τ_w and x. The results are important in indicating general trends in $f_1''(\xi, 0)$.

Consider first the smooth curves for S_{L2} which pass through the origin in all cases. The behaviour for both hot and cold walls is similar to that for $S_w = 0$ and $M_\infty = 0$ (the incompressible case), suggesting that the familiar square-root behaviour of the skin friction is reproduced.

H_L also describes smooth curves in all cases with heat transfer. It appears that $g_1'(0, 0) =$ constant; that is, the heat transfer does not vanish at separation.

The curves for S_{L1} shed further light on the behaviour of $f_1''(\xi, 0)$. The curves have not been continued to $\xi = 0$ because the results are imprecise there. However, the general trend of S_{L1} near $\xi = 0$ can be seen from tables 1 and 2. For a cold wall ($S_w = -\frac{1}{2}$) we note that S_{L1} decreases at first as $\xi \to 0$ and near $\xi = 0$ begins to increase. For a hot wall ($S_w = 1$), S_{L1} decreases as $\xi \to 0$ and the rate of decrease becomes more rapid as the origin is approached. When $S_w = 0$ and $M_\infty = 0$, S_{L1} decreases steadily as $\xi \to 0$.

With the above results in mind the solutions obtained by Stewartson (1962) and Buckmaster (1970) may be considered. The expansions for further f_n and g_n are lengthy and for further details the reader is referred to the original papers. In Buckmaster's notation the leading terms of the skin friction are

$$\xi^2(2\alpha_{10}\log\xi + 2\alpha_{11} + \dots), \tag{38}$$

where

$$\alpha_{10} = \frac{-B_1\, 2\pi^{\frac{1}{2}}(-\frac{1}{4})!}{64(\frac{1}{4}!)^3}, \quad g_1 = B_1\eta. \tag{39}$$

Buckmaster reasoned that, as the skin friction is positive just prior to separation, α_{10} must be negative. Thus $B_1 > 0$, which implies that the wall is cold.

In the incompressible case it is possible to match the numerical results with the expansions for the skin friction and velocity profile at separation. This determines a constant α_1 which corresponds to the constant α_{11} for the compressible case (see Jones 1948; Leigh 1955; Terrill 1960). The expansions are of practical use to fourth order for the skin friction and to fifth order for the velocity profile. In the compressible case the expansions are unwieldy to use in practice and only a rough match can really be attempted. Furthermore, it is almost impossible to match the velocity profile as the presence of logarithms in f_1 means that the variables ξ and η cannot be combined to permit a transformation back into Y'; neither can we set $\xi = 0$. A small value of ξ could be chosen but then the variable η becomes large, making the evaluation of functions such as $g_2(\eta)$ impractical.

With the above reservations, an attempt has been made to match the skin friction and heat transfer with the values obtained from numerical results. We use

$$f_1''(\xi, 0) = 2\alpha_{10}\log\xi + 2\alpha_{11} + 2\alpha_{12}\log|\log\xi| + 2\alpha_{13}\log|\log\xi|/\log\xi + \dots,$$

$$g_1'(\xi, 0) = B_1,$$

$$g_2'(\xi, 0) = -\bar{g}_2'(0)\, B_1(2\alpha_{10}\log\xi + 2\alpha_{11} + 2\alpha_{12}\log|\log\xi| + 2\alpha_{13}\log|\log\xi|/\log\xi + \dots),$$

Compressible flow past a circular cylinder

M_∞	S_w	α_{11}	B_1
0	0	0·67	0
0	$-\frac{1}{2}$	0·72	0·395
0	1	0·64	−0·161
0·4	$-\frac{1}{2}$	0·88	0·444
0·4	1	0·86	−0·170
	Linearly retarded mainstream		
0	0	0·51	0
0	$-\frac{1}{2}$	0·60	0·583
0	1	0·48	−0·297
1	$-\frac{1}{2}$	0·52	0·543
1	1	0·44	−0·288

TABLE 3

where $\alpha_{10} = (-0{\cdot}091148)B_1$, $\alpha_{12} = (-0{\cdot}386294)\alpha_{10}$, $\alpha_{13} = (0{\cdot}149223)\alpha_{10}$ and $\bar{g}_2'(0) = -1{\cdot}111552$. Hence

$$S_{L2} = \xi(2\alpha_{10}\log\xi + 2\alpha_{11} + 2\alpha_{12}\log|\log\xi| + 2\alpha_{13}\log|\log\xi|/\log\xi + \ldots) \tag{40}$$

and

$$H_L = B_1 - \xi\bar{g}_2'(0)B_1(2\alpha_{10}\log\xi + 2\alpha_{11} + 2\alpha_{12}\log|\log\xi| + 2\alpha_{13}\log|\log\xi|/\log\xi + \ldots) + \ldots). \tag{41}$$

The right-hand sides of the above equations have been evaluated for various combinations of α_{11} and B_1 near $\xi = 0{\cdot}01$ and the results have been compared with those for S_{L2} and H_L obtained numerically. The use of (40) and (41) has not been restricted to the cold-wall case and it is the results for the hot wall which are important. In table 3 we list the values α_{11} and B_1 for the specific cases considered.

For both α_{11} and B_1 the last figure is unreliable. However, using the fuller series in the incompressible case ($M_\infty = 0$, $S_w = 0$) the values obtained for the constant α_1 were as follows: for incompressible flow past a cylinder $\alpha_1 = 0{\cdot}673$ (Terrill (1960) found $\alpha_1 = 0{\cdot}677$); for flow with a linearly retarded mainstream $\alpha_1 = 0{\cdot}502$ (Leigh (1955) found $\alpha_1 = 0{\cdot}492$). No difficulty was encountered in using the expansions for a hot wall. The crux of the matter is: when does (38) become negative? From (39), α_{10} is approximately $0{\cdot}1B_1$ and $|B_1|$ can be seen to be smaller than α_{11} for a hot wall. Before (38) becomes negative ξ must be very small ($< 10^{-6}$). But $x_s - x = l_s\xi^4$ and the distance between x and separation before (38) becomes negative is extremely small.

The results for S_{L1} also seem to indicate that the expansion is still valid for a hot wall with the appropriate logarithmic behaviour. We noted that, as $\xi \to 0$, S_{L1} decreased more rapidly, which is due to the effect of the logarithmic term as $\xi \to 0$. Conversely, for a cold wall the logarithmic term would tend to increase S_{L1} as $\xi \to 0$. Nevertheless, it does appear from the figures that the curve for S_{L1} will cross the ξ axis before $\xi = 0$ in the hot-wall case. It is impossible to determine this absolutely since the computation of more accurate values of S_{L1} close to $\xi = 0$ requires that τ_w be calculated accurately to more decimal places. Also the value of x_s, which gives the origin for ξ, would need to be calculated to more decimal places. This whole process would require the use of double-precision arithmetic and some thought is currently being given to attempting this. With this reservation in mind, however, it does appear from the graphs that the skin friction behaves, close to $\xi = 0$, in the manner suggested by the Buckmaster expansion for both hot and cold walls.

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